Two-loop renormalization of the effective field theory of a static quark

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Abstract We give a recurrence relation for two-loop integrals encountered in the effective field theory of an infinitely heavy quark, Q, interacting with gluons and $N_{\rm L}$ massless quarks, q, from which we obtain exact two-loop results, in any dimension and covariant gauge, for the propagator of Q and the vertex function of the heavy-light current $J \equiv \overline{Q}\Gamma q$, at zero q-momentum. The anomalous dimension of the Q-field agrees with the recent result of Broadhurst, Gray and Schilcher. The anomalous dimension of the current is

$$\widetilde{\gamma}_{\rm J} \equiv \frac{{\rm d}\log \widetilde{Z}_{\rm J}}{{\rm d}\log\mu} = -\frac{\alpha_{\rm s}}{\pi}\left\{1 + \left(\frac{127+56\zeta(2)-10N_{\rm L}}{72}\right)\frac{\alpha_{\rm s}}{\pi} + O(\alpha_{\rm s}^2)\right\}$$

which gives the new two-loop correction to the result of Voloshin and Shifman.

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1 Introduction

Recently an interesting new approach to QCD problems involving a heavy quark was proposed, namely the effective field theory (EFT) of a static quark [1-5]. (See also the review [6].) To leading order in the $1/m$ expansion, the EFT lagrangian is [1]

$$L = Q_0^\dagger \mathrm{i} D_0 Q_0 + \overline{q}_0 \mathrm{i} \not{D} q_0 - \tfrac{1}{4} G^a_{0\mu\nu} G^a_{0\mu\nu} - \left(\partial_\mu A^a_{0\mu}\right)^2 / 2a_0 + (\text{ghost term}) \qquad (1)$$

where q_0 and $A_{0\mu}$ are the bare light-quark and gluon fields of conventional QCD and Q_0 is the bare static-quark field, which is a two-component spinor. (When used as a 4-component spinor, its lower components are assumed to vanish.) The free-field propagator of Q_0 is $1/\omega$ (equivalently, $(1+\gamma_0)/2\omega$, in 4-component form) where $\omega \equiv p_0 - m$ is the energy by which it is off-shell. Its coupling to gluons is given by the vertex $\mathrm{i}g_0 v_\mu t^a$, where $v_\mu = (1, \mathbf{0})$ is its velocity. The lagrangian (1) possess an SU(2) static-quark spin symmetry [4].

In this paper we consider the two-loop renormalization of EFT, using dimensional regularization in $D \equiv 4-2\epsilon$ dimensions. In the $\overline{\mathrm{MS}}$ scheme the bare quantities in (1) are related to the corresponding renormalized quantities by

$$Q_0 = \bar{\mu}^{-\epsilon} \widetilde{Z}_2^{1/2} Q, \ q_0 = \bar{\mu}^{-\epsilon} Z_2^{1/2} q, \ A^a_{0\mu} = \bar{\mu}^{-\epsilon} Z_3^{1/2} A^a_\mu, \ g_0 = \bar{\mu}^{\epsilon} Z_\alpha^{1/2} g, \ a_0 = Z_3 a$$

where Z_2, Z_3 and Z_α are the same as in QCD [7], with N_{L} light-quark flavours, and $\bar{\mu}^2 = \mu^2 \mathrm{e}^\gamma / 4\pi$. Here we calculate, to two loops, the static-quark wave-function renormalization constant $\widetilde{Z}_2$ and the renormalization constant $\widetilde{Z}_{\mathrm{J}}$ of the heavy-light current $J_0 = \overline{Q}_0 \Gamma q_0 = \widetilde{Z}_{\mathrm{J}} J$, where Γ is some (irrelevant) gamma matrix. For $\widetilde{Z}_2$ we obtain the same result as Broadhurst, Gray and Schilcher [8], who have independently extracted it from the gauge-invariant singularities of on-shell wave-function renormalization of a finite-mass quark [9]. ¿From $\widetilde{Z}_{\mathrm{J}}$ we obtain the two-loop correction to the leading-order result [10, 11] for the anomalous dimension $\widetilde{\gamma}_{\mathrm{J}} \equiv \mathrm{d} \log \widetilde{Z}_{\mathrm{J}} / \mathrm{d} \log \mu$. This is needed for the renormalization-group analysis of the extrapolations of lattice computations of heavy-meson decay constants [12].

Renormalization of EFT corresponds closely to renormalization of the Wilson line [13-17], which coincides with the static-quark propagator in coordinate space. In particular the lagrangian (1) is essentially the same as that proposed in [14], for investigating Wilson lines, and used in the two-loop calculations of [16, 17]. In [16] Aoyama calculated $\widetilde{Z}_2$ in the Feynman gauge, at $N_{\mathrm{L}} = 0$; we disagree with his result. In [17] Korchemsky and Radyushkin calculated the two-loop anomalous dimension of a heavy-heavy current, which depends on the relative velocity of initial and final quarks; the corresponding one-loop result [13] has recently been rediscovered in EFT [5].

2 One- and two-loop integrals

The static-quark bare self-energy term $-\mathrm{i}\Sigma(\omega)$ is given, to two loops, by the diagrams of Fig 1, where the blob in Fig 1a includes the light-quark, gluon and ghost loops in the gluon propagator. We take the light quarks as massless and easily evaluate the master one-loop EFT integral

$$\int \mathrm{d}^D k \left(\frac{-1}{k^2}\right)^\alpha \left(\frac{\omega}{\omega + k_0}\right)^p \equiv \mathrm{i}\pi^{D/2} (-2\omega)^{(D-2\alpha)} I(\alpha, p)$$

using the parameterization

$$\frac{1}{A^\alpha P^p} = \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)\Gamma(p)} \int_0^\infty \frac{y^{p-1}\mathrm{d}y}{(A+yP)^{\alpha+p}}$$

where we adopt the convention that massless lines have greek indices and static lines have roman ones. We find that

$$I(\alpha,p) = \frac{\Gamma(2\alpha+p-D)\Gamma(D/2-\alpha)}{\Gamma(\alpha)\Gamma(p)} \tag{2}$$

which enables us to evaluate diagrams of the form of Fig 1a.

The master two-loop EFT integral, for diagrams of the form of Fig 1b, is

$$\int \mathrm{d}^D k \int \mathrm{d}^D l \left(\frac{-1}{k^2}\right)^\alpha \left(\frac{-1}{l^2}\right)^\beta \left(\frac{-1}{(k-l)^2}\right)^\gamma \left(\frac{\omega}{\omega+k_0}\right)^p \left(\frac{\omega}{\omega+l_0}\right)^q$$
$$\equiv -\pi^D(-2\omega)^{2(D-\alpha-\beta-\gamma)} I(\alpha,\beta,\gamma,p,q)$$

In degenerate cases one uses (2) and the corresponding result [18] for the master one-loop diagram of massless QCD

$$G(\alpha,\beta) = \frac{\Gamma(\alpha+\beta-D/2)\Gamma(D/2-\alpha)\Gamma(D/2-\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(D-\alpha-\beta)}$$

to obtain

$$\begin{aligned} I(\alpha,\beta,0,p,q) &= I(\alpha,p)I(\beta,q) && (3)\\ I(\alpha,0,\gamma,p,q) &= I(\alpha,2\gamma+p+q-D)I(\gamma,q) && (4)\\ I(\alpha,\beta,\gamma,p,0) &= I(\alpha+\beta+\gamma-D/2,p)G(\beta,\gamma) && (5)\end{aligned}$$

which suffice for diagrams of the form of Fig 1c and Fig 1d, since the integrand of Fig 1c has only 4 factors in the denominator and the integrand of Fig 1d can be expressed as the sum of three such terms after multiplying by

$$1 = \frac{1}{\omega}\left\{(\omega+k_0)+(\omega+l_0)-(\omega+k_0+l_0)\right\}$$

This is an enormous simplification in comparison with on-shell massive QCD calculations [8, 9], where this last diagram was the most difficult.

In general we use a recurrence relation to reduce the non-degenerate case of the master integral to a sum of known degenerate cases. Using the method of integration by parts [9, 18], we have found that

$$(D-\alpha-2\gamma-p-q+1)I = \left\{\alpha\mathbf{A}^+(\mathbf{\Gamma}^- - \mathbf{B}^-) + (2(D-\alpha-\beta-\gamma)-p-q+1)\mathbf{Q}^-\right\} I \tag{6}$$

where, for example, $\mathbf{A}^+$ raises α and $\mathbf{Q}^-$ lowers q. Again this is simpler than the triangle relation [18] of massless QCD, because no more than three terms appear and no static-line index is increased.

Now it is a matter of programming to reduce every one-loop integral to a rational function of $D \equiv 4-2\epsilon$ times

$$\Gamma_1 \equiv (-2\omega)^{-2\epsilon}\Gamma(-\epsilon)\Gamma(1+2\epsilon)/(4\pi)^{D/2} \tag{7}$$

and every two-loop integral from Fig 1 to a linear combination of Γ_1^2 and

$$\begin{aligned} \Gamma_2 &\equiv (-2\omega)^{-4\epsilon}\Gamma^2(-\epsilon)\Gamma(1+4\epsilon)/(4\pi)^D && (8) \\ &= (1+4\zeta(2)\epsilon^2-16\zeta(3)\epsilon^3)\Gamma_1^2+0(\epsilon^2) && (9) \end{aligned}$$

with coefficients which are rational functions of D. We used REDUCE [19] on a Vax and checked results with Mathematica [20] on a Macintosh. The programming effort and run times were far less than in the related QCD calculations [8, 9].

3 Anomalous dimension of the static-quark field

Using these methods, it is straightforward to compute the two-loop corrections to the bare propagator $S_0(\omega)$. We find that

$$\begin{aligned} \omega S_0(\omega) &= \frac{\omega}{\omega-\Sigma(\omega)} = 1+Z_1\Gamma_1 C_{\rm F} g_0^2+\Big\{Z_{\rm A1}C_{\rm A}\Gamma_1^2 \\ &\qquad +(Z_{\rm A2}C_{\rm A}+Z_{\rm F2}C_{\rm F}+Z_{\rm L2}T_{\rm F}N_{\rm L})\Gamma_2\Big\}C_{\rm F}g_0^4+O(g_0^6) \qquad (10) \end{aligned}$$

with coefficients

$$\begin{aligned} Z_1 &= a_0-\frac{D-1}{D-3} \\ Z_{\rm A1} &= -\frac{Z_1}{D-3} \\ Z_{\rm A2} &= \frac{(D-2)^2(D-5)}{2(2D-7)(D-3)^3(D-6)}+\frac{(D^2-4D+5)Z_1}{2(D-3)^2(D-6)}-\frac{(D^2-9D+16)Z_1^2}{8(D-3)(D-6)} \\ Z_{\rm F2} &= \frac{Z_1^2}{2} \\ Z_{\rm L2} &= \frac{2(D-2)}{(2D-7)(D-3)(D-6)} \end{aligned}$$

and colour factors $C_{\rm A}=N$, $C_{\rm F}=(N^2-1)/2N$, $T_{\rm F}=\frac{1}{2}$, for an SU(N) gauge group, or $C_{\rm A}=0$, $C_{\rm F}=T_{\rm F}=1$ for a U(1) gauge group.

We now renormalize the bare static-quark field $Q_0=\bar{\mu}^{-\epsilon}\widetilde{Z}_2^{1/2}Q$ using the minimal wave-function renormalization constant $\widetilde{Z}_2$ which makes the renormalized propagator $S(\omega)=S_0(\omega)/\widetilde{Z}_2$ finite. (Tildes are used here to distinguish EFT renormalization constants from those of QCD.) We work in the $\overline{\rm MS}$ scheme, in which the bare coupling and gauge parameter are renormalized by [7]

$$\begin{aligned} \frac{g_0^2}{g^2\bar{\mu}^{2\epsilon}} &= Z_\alpha = 1-\frac{\alpha_{\rm s}}{4\pi\epsilon}\left(\frac{11}{3}C_{\rm A}-\frac{4}{3}T_{\rm F}N_{\rm L}\right)+O(\alpha_{\rm s}^2) && (11) \\ \frac{a_0}{a} &= Z_3 = 1+\frac{\alpha_{\rm s}}{4\pi\epsilon}\left(\frac{13-3a}{6}C_{\rm A}-\frac{4}{3}T_{\rm F}N_{\rm L}\right)+O(\alpha_{\rm s}^2) && (12) \end{aligned}$$

where $\alpha_{\rm s}\equiv g^2/4\pi$. We posit a minimal form $\widetilde{Z}_2=1+(C_{11}(a)/\epsilon)\alpha_{\rm s}/\pi+(C_{22}(a)/\epsilon^2+C_{21}(a)/\epsilon)\alpha_{\rm s}^2/\pi^2+O(\alpha_{\rm s}^3)$ and determine its coefficients from (10). The renormalizability of the theory requires that this constant make $S_0(\omega)/\widetilde{Z}_2$ finite for all ω, which provides a highly non-trivial check on our result. Then, using the μ dependence of $\alpha_{\rm s}$ and a, implied by (11,12), we obtain the anomalous dimension $\widetilde{\gamma}_{\rm F}\equiv {\rm d}\log\widetilde{Z}_2/{\rm d}\log\mu$ of the static Q field. As a check on the whole procedure we mirror it, step by step,

by that for a massless quark, to obtain the corresponding QCD quantity $\gamma_{\rm F}$ for the massless q field from the recurrence relations of [18]. Our result is

$$\begin{aligned}\widetilde{\gamma}_{\rm F} &= \frac{(a-3)C_{\rm F}\alpha_{\rm s}}{2\pi}+\left\{\left(\frac{a^2}{32}+\frac{a}{4}-\frac{179}{96}\right)C_{\rm A}+\frac{2}{3}T_{\rm F}N_{\rm L}\right\}\frac{C_{\rm F}\alpha_{\rm s}^2}{\pi^2}+O(\alpha_{\rm s}^3)\\ &= \gamma_{\rm F}-\frac{3C_{\rm F}\alpha_{\rm s}}{2\pi}-\left\{\frac{127}{48}C_{\rm A}-\frac{3}{16}C_{\rm F}-\frac{11}{12}T_{\rm F}N_{\rm L}\right\}\frac{C_{\rm F}\alpha_{\rm s}^2}{\pi^2}+O(\alpha_{\rm s}^3) \qquad (13)\end{aligned}$$

where the latter form was independently obtained in [8] from the gauge-invariant singularities of *on*-shell wave-function renormalization of a *finite*-mass quark. Note that (13) vanishes in QED in the renormalized Yennie gauge [21] $a=3$, for which there is no 'infrared catastrophe' [22]. From (13) we find that $\gamma_{\rm F}$ agrees with the QCD result of Egorian and Tarasov [23] (misquoted in [7], which contains sign errors).

Setting $a=1$ and $N_{\rm L}=0$ in (13), we discover an error in [16], where the renormalization of the Wilson line was calculated (neglecting light-quark loops) in the Feynman gauge (where Fig 1b gives no contribution). The $\alpha_{\rm s}^2/\epsilon^2$ terms in [16] are inconsistent with the renormalization group and the $\alpha_{\rm s}^2/\epsilon$ terms include the contributions of Figs 1c,d with the wrong sign. This incorrect result was used in [24], where the authors also omitted a factor of 2 from the two-loop term in the anomalous dimension. In view of these approximations and errors the two-loop analysis of [24] needs reconsideration.

4 Anomalous dimension of the heavy-light current

The simplest method for finding the anomalous dimension $\widetilde{\gamma}_{\rm J}$ of the current $J_0=\overline{Q}_0\Gamma q_0$ is to consider the bare vertex function with zero light-quark momentum. The diagrams of Fig 2 then differ from those of Fig 1 only in their numerator structure and the indices of their denominators. The effect is to multiply the gamma matrix Γ by a factor $\Gamma_0(\omega)$, which is gauge-dependent and more complicated than (10). But it becomes much simpler when one multiplies by the external static-quark bare propagator $S_0(\omega)$. We find that

$$\begin{aligned}V_0(\omega) \equiv \omega S_0(\omega)\Gamma_0(\omega) &= 1+V_1\Gamma_1C_{\rm F}g_0^2+\Big\{(V_{\rm A1}C_{\rm A}+V_{\rm F1}C_{\rm F})\Gamma_1^2\\ &\quad+(V_{\rm A2}C_{\rm A}+V_{\rm F2}C_{\rm F}+V_{\rm L2}T_{\rm F}N_{\rm L})\Gamma_2\Big\}C_{\rm F}g_0^4+O(g_0^6) \qquad (14)\end{aligned}$$

is gauge-invariant. The coefficients are

$$\begin{aligned}V_1 &= -\frac{D-1}{D-3}\\ V_{\rm A1} &= \frac{D^2-4D-1}{2(D-3)^2(D-4)}\\ V_{\rm F1} &= \frac{D-2}{(D-3)(D-4)}\\ V_{\rm A2} &= -\frac{2D^4-22D^3+89D^2-151D+78}{2(2D-7)(D-3)^2(D-4)(D-6)}\\ V_{\rm F2} &= -\frac{(D^2-5D+2)D}{2(D-3)^2(D-4)(D-6)}\\ V_{\rm L2} &= \frac{2(D-2)}{(2D-7)(D-3)(D-6)}\end{aligned}$$

independently of the spin structure of the current J.

The renormalization constant $\widetilde{Z}_{\rm J} = \widetilde{Z}_\Gamma \widetilde{Z}_2^{1/2} Z_2^{1/2}$ may be calculated from the requirement that the renormalized vertex function $\Gamma(\omega) = \Gamma_0(\omega)/\widetilde{Z}_\Gamma$ be finite. Equivalently, but more conveniently, we have $\widetilde{Z}_{\rm J} = \widetilde{Z}_V \widetilde{Z}_2^{-1/2} Z_2^{1/2}$, where $\widetilde{Z}_{\rm V} = \widetilde{Z}_\Gamma \widetilde{Z}_2$ makes $V(\omega) = V_0(\omega)/\widetilde{Z}_{\rm V}$ finite. This gives two highly non-trivial checks on our results. First, the $\alpha_{\rm s}^2/\epsilon^2$ terms in (14) must be such that division by a constant makes $V(\omega)$ finite for all ω. Second, the anomalous dimension $\widetilde{\gamma}_{\rm V} = {\rm d}\log\widetilde{Z}_{\rm V}/{\rm d}\log\mu$ must be gauge-invariant, to ensure that $\widetilde{\gamma}_{\rm J} = \widetilde{\gamma}_{\rm V} - \frac{1}{2}\widetilde{\gamma}_{\rm F} + \frac{1}{2}\gamma_{\rm F}$ is also gauge-invariant. As a third check, we mirrored this EFT calculation, step by step, by the QCD calculation of $\gamma_{\overline{q}q}$, obtaining agreement with Tarrach [25]. These checks give us considerable confidence in our final result

$$\begin{aligned}
\widetilde{\gamma}_{\rm J} &= -\frac{3\alpha_{\rm s}C_{\rm F}}{4\pi} - \left\{\frac{49}{96}C_{\rm A} - \frac{5}{32}C_{\rm F} - \frac{5}{24}T_{\rm F}N_{\rm L} - (\tfrac{1}{4}C_{\rm A} - C_{\rm F})\zeta(2)\right\}\frac{C_{\rm F}\alpha_{\rm s}^2}{\pi^2} + O(\alpha_{\rm s}^3) \\
&= \tfrac{1}{2}\gamma_{\overline{q}q} + \left\{\tfrac{1}{2}C_{\rm A} + \tfrac{1}{4}C_{\rm F} + \left(\tfrac{1}{4}C_{\rm A} - C_{\rm F}\right)\zeta(2)\right\}\frac{C_{\rm F}\alpha_{\rm s}^2}{\pi^2} + O(\alpha_{\rm s}^3) \qquad (15)
\end{aligned}$$

whose second form we have also verified from the quark-condensate contributions to the correlator of J [26].

5 Summary and conclusions

We have given a method for calculating one-scale one- and two-loop EFT integrals. The master two-loop integral can be reduced by the recurrence relation (6) to the degenerate cases (3,4,5), which yield the structures (7,8). We have used this method to calculate the static-quark propagator and a particular case of the heavy-light vertex function, obtaining (10,14) in an arbitrary spacetime dimension and in any covariant gauge. From these follow the anomalous dimensions of the static-quark propagator (13) and the heavy-light current (15). The first agrees with [8] and contradicts an old result for the Wilson line [16]. The second is also confirmed in [26]. It gives the radiative correction to the one-loop result of [10, 11] and is necessary for matching QCD and EFT currents, with account of the radiative corrections of [1]. The lack of it in [12] limited the accuracy of extrapolation of lattice computation.

Novel features of EFT calculations include: the trivial simplification of diagrams such as Figs 1d and 2i, which are the most difficult in on-shell massive QCD calculations [8, 9]; an algorithm more efficient than the triangle relation [18] of massless QCD; the appearance of $\zeta(2) = \pi^2/6$ in counterterms, via the relation (9) between the basic integral structures, in contrast to the situation in QCD, where $\zeta(3)$ is the first term to appear in the expansion.

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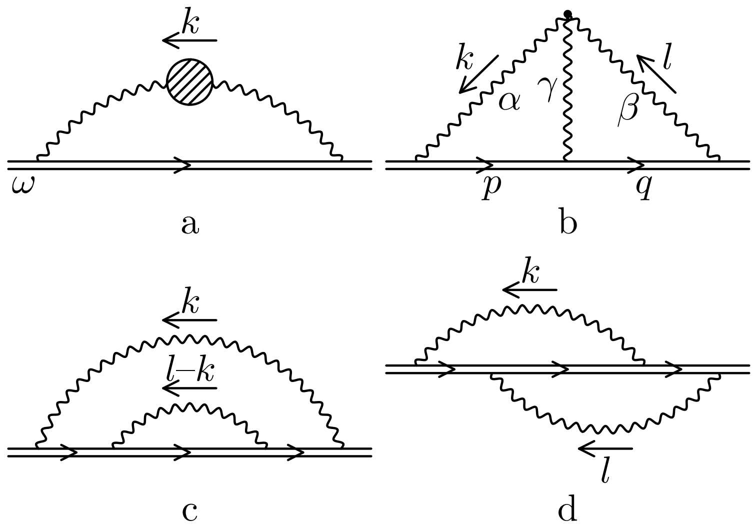

Figure 1: Heavy-quark self-energy diagrams, to two loops

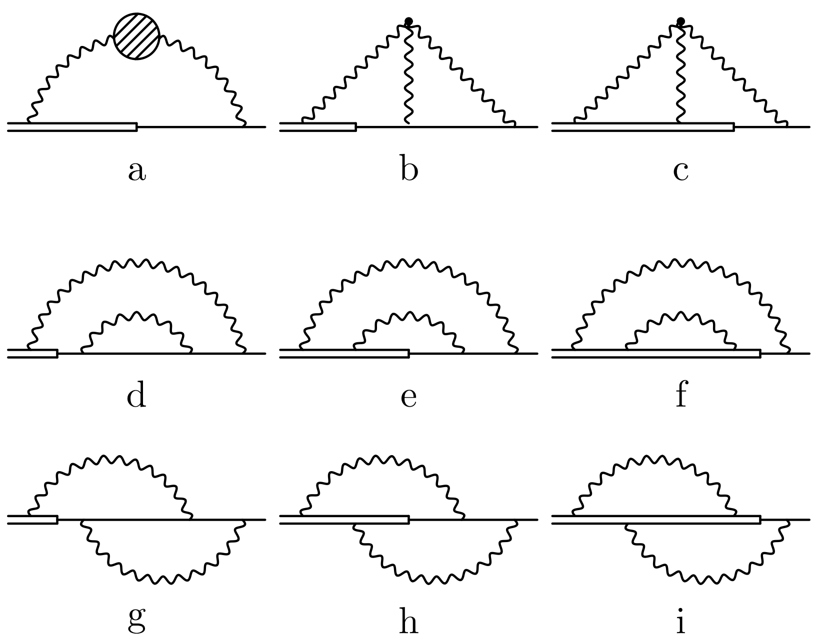

Figure 2: Heavy-light proper vertex diagrams, to two loops